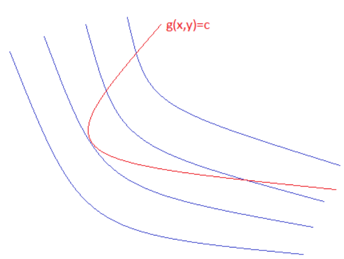
Lagrange Multiplier (Part II)

Liming Pang

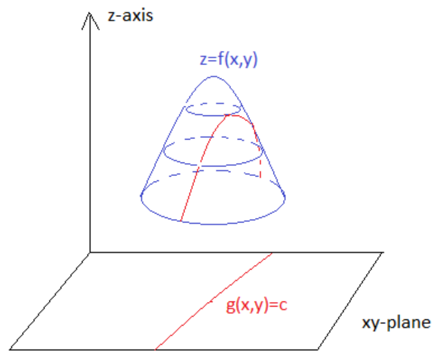
1 Why the Method of Lagrange Multiplier Works

Now let's explain why the method of Lagrange multiplier works. We will study the situation for optimizing a to variable function $f(x, y)$ under the constraint $g(x, y) = c$. The cases with more variables or more constraints can be shown in the similar fashion.

If (x_0, y_0) is a maximum/minimum point of $f(x, y)$ under the constraint $g(x, y) = c$, we need to show that $\nabla f(x_0, y_0)$ is parallel to $\nabla g(x_0, y_0)$. Suppose they are not parallel. Since gradient is normal to level sets, we see that the assumption implies the level set $f(x, y) = f(x_0, y_0)$ and the level set $g(x, y) = c$ are not tangent at the intersection (x_0, y_0). Then it means on each side of $f(x, y) = f(x_0, y_0)$, there are points on $g(x, y) = c$, and for points on one of the two sides, the values under f is larger than $f(x_0, y_0)$. This can be shown as in the following figure. The following figure shows what's going on.



We can also draw a 3-dimensional figure to illustrate:



Algebraically, we can apply the Implicit Function Theorem. By assumption, $\nabla g(x_0, y_0) = (\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0)) \neq \vec{0}$, we may assume $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$. This condition implies $g(x, y) = c$ implicitly defines a function $y = h(x)$ near (x_0, y_0) such that $g(x, h(x)) = 0$ and $y_0 = h(x_0)$.

By Implicit Function Theorem, we know that $h'(x_0) = -\frac{\frac{\partial g}{\partial x}(x_0,y_0)}{\frac{\partial g}{\partial y}(x_0,y_0)}$. Define $\phi(x) = f(x, h(x))$, then the assumption that (x_0, y_0) is an extreme point for the constraint optimization indicates that x_0 is an extreme point for $\phi(x)$, so $\phi'(x) = 0$.

By the Chain Rule, we see

$$0 = \phi'(x) = \frac{\partial f}{\partial x}(x_0, y_0) + \frac{\partial f}{\partial y}(x_0, y_0)h'(x_0) = \frac{\partial f}{\partial x}(x_0, y_0) - \frac{\partial f}{\partial y}(x_0, y_0)\frac{\frac{\partial g}{\partial x}(x_0, y_0)}{\frac{\partial g}{\partial y}(x_0, y_0)}$$

i.e.

$$\frac{\partial f}{\partial x}(x_0, y_0)\frac{\partial g}{\partial y}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0)\frac{\partial g}{\partial x}(x_0, y_0)$$

This implies $\nabla f(x_0, y_0)$ is parallel to $\nabla g(x_0, y_0)$, since if $\frac{\partial g}{\partial x}(x_0, y_0) \neq 0$, we see $\frac{\frac{\partial f}{\partial x}(x_0,y_0)}{\frac{\partial g}{\partial x}(x_0,y_0)} = \frac{\frac{\partial f}{\partial y}(x_0,y_0)}{\frac{\partial g}{\partial y}(x_0,y_0)}$. Let $\lambda \neq 0$ be this ratio; if $\frac{\partial g}{\partial x}(x_0, y_0) = 0$, we get $\frac{\partial f}{\partial x}(x_0, y_0) = 0$ as well, this is the case for $\lambda = 0$.

2 Interpretation of the Lagrange Multiplier

In this section we will explore what information the Lagrange multiplier λ can bring to us.

Assume (x_0, y_0) is a maximal point for $f(x, y)$ subject to constraint $g(x, y) = c$. We thus know:

$$\begin{cases} \nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \\ g(x_0, y_0) = c \end{cases}$$

Now we would like to know how the constraint optimal value changes if we change the constraint constant from c by a small change to some c'.

Recall that given a function $f(x, y)$, there is a differential

$$df = \frac{\partial f}{\partial x}(x_0, y_0)dx + \frac{\partial f}{\partial y}(x_0, y_0)dy$$

which gives a good estimation of how the value of the function changes if x, y are changed by small amount dx, dy respectively from x_0, y_0.

Now assume the new extreme point of f subject to the constraint c' is obtained from (x_0, y_0) by a change of (dx, dy), we get the corresponding change in the value of f to be

$$\begin{aligned} df &= \frac{\partial f}{\partial x}(x_0, y_0)dx + \frac{\partial f}{\partial y}(x_0, y_0)dy \\ &= (\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0)) \cdot (dx, dy) \\ &= \lambda(\frac{\partial g}{\partial x}(x_0, y_0), \frac{\partial g}{\partial y}(x_0, y_0)) \cdot (dx, dy) \\ &= \lambda(\frac{\partial g}{\partial x}(x_0, y_0)dx + \frac{\partial g}{\partial y}(x_0, y_0)dy) \\ &= \lambda dg \\ &= \lambda(c' - c) \end{aligned}$$

The above computation leads to the following theorem:

Theorem 1. *If the constraint is changed by a small amount from $g(x, y) = c$ to $g(x, y) = c'$, then the optimal value will change by about $\lambda(c' - c)$, where λ is the Lagrange multiplier.*

Another way of explanation is that if we denote P to be the optimal value of $f(x,y)$ subject to the constraint $g(x,y)=c$, then P can be regarded as a function of c, i.e. $P=P(c)$, then the above argument indicates $\lambda=\frac{dP}{dc}$.

Example 2. *There is an economic understanding of the above discussion:*

A company is producing two brands, A and B. When producing x units of A and y units of B, the cost is $g(x,y)$ dollars and the profit is $f(x,y)$ dollars. Currently the company inputs c dollars everyday, and the daily profit is P (i.e. the maximum value for $f(x,y)$ subject to the constraint $g(x,y)=c$ is P). If we know at this point the Lagrange multiplier λ, then if the daily input increases by 1 dollar, the daily profit will increase by about λ dollars.

*In economics, the Lagrange multiplier is called the **shadow price** of the resource. It is the marginal profit with respect to the budget.*

Example 3. *Now let's compute a concrete example to see how well the Lagrange multiplier estimates the actual change of optimal value. We are revisiting Example 4 in Part I:*

A person has utility function $u(x,y)=10xy+5x+2y$. Suppose the price for one unit of x is 2 dollars and the price for one unit of y is 5 dollars. If the person has 100 dollars that can be spent on x and y, find x and y that maximize the utility.

The question is to maximize $u(x,y)=10xy+5x+2y$ under the constraint $2x+5y=100$.

Let $g(x,y)=2x+5y$, by the Lagrange method, we have

$$\begin{cases}\nabla u(x,y)=\lambda\nabla g(x,y)\\ g(x,y)=100\end{cases}$$

i.e.

$$\begin{cases}10y+5=\lambda\times 2\\ 10x+2=\lambda\times 5\\ 2x+5y=100\end{cases}$$

We get $\lambda=51.45, x=25.525, y=9.79$, the maximal utility is $u(25.525,9.79)=2646.1025$

Now assume the person has 1 more dollar in budget, we have

$$\begin{cases}\nabla u(x,y)=\lambda\nabla g(x,y)\\ g(x,y)=101\end{cases}$$

i.e.

$$\begin{cases} 10y + 5 = \lambda \times 2 \\ 10x + 2 = \lambda \times 5 \\ 2x + 5y = 101 \end{cases}$$

We get $\lambda = 51.95, x = 25.7755, y = 9.89$*, the maximal utility is* $u(25.7755, 9.89) = 2697.85445$

So the actual increase of maximal utility is $2697.85445 - 2646.1025 = 51.75195$*, and the initial Lagrange multiplier for budget being* 100 *dollars is* 51.45*, which is close to the actual increase.*